CHAPTER 8

Algebraic Expressions and Identities

0852CH09

8.1 Addition and Subtraction of Algebraic Expressions

In earlier classes, we have already become familiar with what algebraic expressions (or simply expressions) are. Examples of expressions are:

$$x + 3,\ 2y - 5,\ 3x^2,\ 4xy + 7 \text{ etc.}$$

In the earlier classes, we have also learnt how to add and subtract algebraic expressions. For example, to add $7x^2 - 4x + 5$ and $9x - 10$, we do

$$\begin{array}{rrrr} & 7x^2 & -\ 4x & +\ 5 \\ + & & 9x & -\ 10 \\ \hline & 7x^2 & +\ 5x & -\ 5 \end{array}$$

Observe how we do the addition. We write each expression to be added in a separate row. While doing so we write like terms one below the other, and add them, as shown. Thus $5 + (-10) = 5 - 10 = -5$. Similarly, $-4x + 9x = (-4 + 9)x = 5x$. Let us take some more examples.

Example 1: Add: $7xy + 5yz - 3zx$, $4yz + 9zx - 4y$, $-3xz + 5x - 2xy$.

Solution: Writing the three expressions in separate rows, with like terms one below the other, we have

$$\begin{array}{rrrrrr} & 7xy & +\ 5yz & -3zx & & \\ + & & 4yz & +\ 9zx & & -\ 4y \\ + & -2xy & & -\ 3zx & +\ 5x & \\ \hline & 5xy & +\ 9yz & +3zx & +\ 5x & -\ 4y \end{array}$$

(Note xz is same as zx)

Thus, the sum of the expressions is $5xy + 9yz + 3zx + 5x - 4y$. Note how the terms, $-4y$ in the second expression and $5x$ in the third expression, are carried over as they are, since they have no like terms in the other expressions.

Example 2: Subtract $5x^2 - 4y^2 + 6y - 3$ from $7x^2 - 4xy + 8y^2 + 5x - 3y$.

Solution:

$$\begin{array}{rrrrrr}
7x^2 & -\,4xy & +\,8y^2 & +\,5x & -\,3y & \\
5x^2 & & -\,4y^2 & & +\,6y & -\,3 \\
\hline
(-) & & (+) & & (-) & (+) \\
2x^2 & -\,4xy & +\,12y^2 & +\,5x & -\,9y & +\,3
\end{array}$$

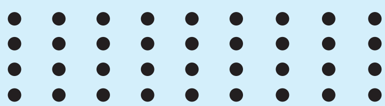
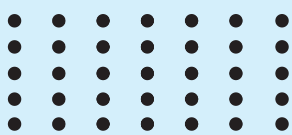
> Note that subtraction of a number is the same as addition of its additive inverse. Thus subtracting -3 is the same as adding $+3$. Similarly, subtracting $6y$ is the same as adding $-6y$; subtracting $-4y^2$ is the same as adding $4y^2$ and so on. The signs in the third row written below each term in the second row help us in knowing which operation has to be performed.

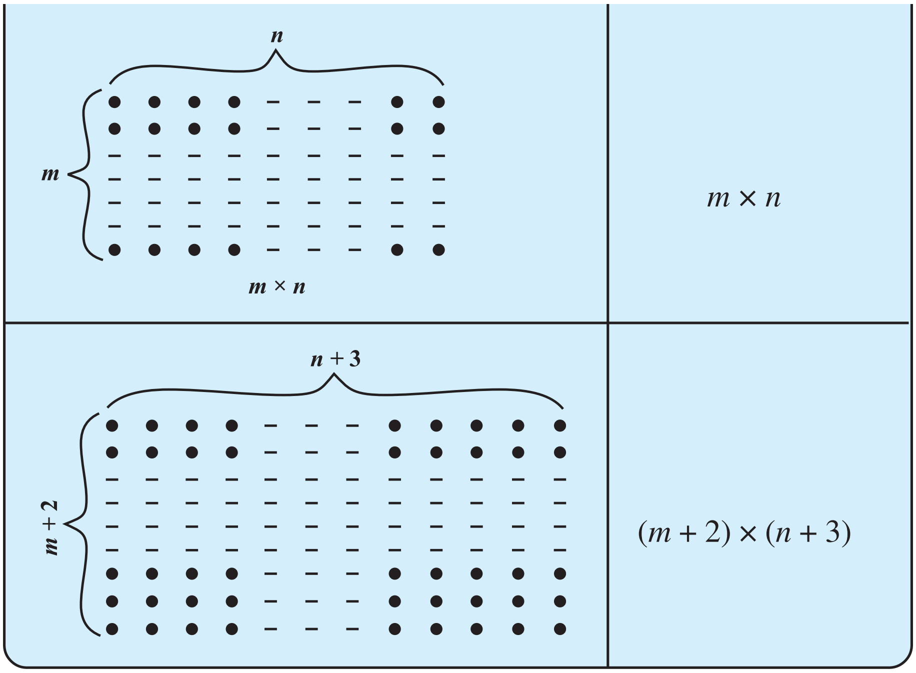
EXERCISE 8.1

1. Add the following.
   (i) $ab - bc$, $bc - ca$, $ca - ab$
   (ii) $a - b + ab$, $b - c + bc$, $c - a + ac$
   (iii) $2p^2q^2 - 3pq + 4$, $5 + 7pq - 3p^2q^2$
   (iv) $l^2 + m^2$, $m^2 + n^2$, $n^2 + l^2$, $2lm + 2mn + 2nl$
2. (a) Subtract $4a - 7ab + 3b + 12$ from $12a - 9ab + 5b - 3$
   (b) Subtract $3xy + 5yz - 7zx$ from $5xy - 2yz - 2zx + 10xyz$
   (c) Subtract $4p^2q - 3pq + 5pq^2 - 8p + 7q - 10$ from $18 - 3p - 11q + 5pq - 2pq^2 + 5p^2q$

8.2 Multiplication of Algebraic Expressions: Introduction

(i) Look at the following patterns of dots.

Pattern of dots	Total number of dots
	4×9
	5×7

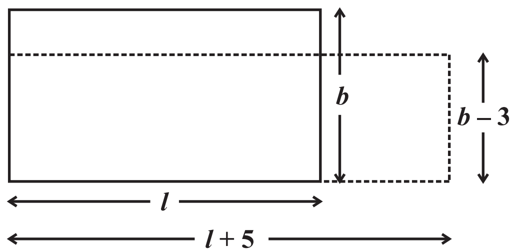


To find the number of dots we have to multiply the expression for the number of rows by the expression for the number of columns.

Here the number of rows is increased by 2, i.e., $m+2$ and number of columns increased by 3, i.e., $n+3$.

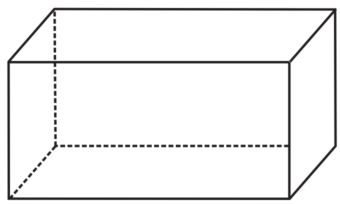
(ii) Can you now think of similar other situations in which two algebraic expressions have to be multiplied?

Ameena gets up. She says, "We can think of area of a rectangle." The area of a rectangle is $l \times b$, where l is the length, and b is breadth. If the length of the rectangle is increased by 5 units, i.e., $(l + 5)$ and breadth is decreased by 3 units , i.e., $(b - 3)$ units, the area of the new rectangle will be $(l + 5) \times (b - 3)$.



To find the area of a rectangle, we have to multiply algebraic expressions like $l \times b$ or $(l + 5) \times (b - 3)$.

(iii) Can you think about volume? (The volume of a rectangular box is given by the product of its length, breadth and height).



(iv) Sarita points out that when we buy things, we have to carry out multiplication. For example, if

price of bananas per dozen = ₹ p

and for the school picnic bananas needed = z dozens,

then we have to pay = ₹ $p \times z$

Suppose, the price per dozen was less by ₹ 2 and the bananas needed were less by 4 dozens.

Then, price of bananas per dozen = ₹ $(p - 2)$

and bananas needed = $(z - 4)$ dozens,

Therefore, we would have to pay = ₹ $(p - 2) \times (z - 4)$

TRY THESE

Can you think of two more such situations, where we may need to multiply algebraic expressions?

[**Hint:**
- Think of speed and time;
- Think of interest to be paid, the principal and the rate of simple interest; etc.]

In all the above examples, we had to carry out multiplication of two or more quantities. If the quantities are given by algebraic expressions, we need to find their product. This means that we should know how to obtain this product. Let us do this systematically. To begin with we shall look at the multiplication of two monomials.

8.3 Multiplying a Monomial by a Monomial

Expression that contains only one term is called a **monomial**.

8.3.1 Multiplying two monomials

We begin with

$4 \times x = x + x + x + x = 4x$ as seen earlier.

Similarly, $4 \times (3x) = 3x + 3x + 3x + 3x = 12x$

Now, observe the following products.

(i) $x \times 3y = x \times 3 \times y = 3 \times x \times y = 3xy$

(ii) $5x \times 3y = 5 \times x \times 3 \times y = 5 \times 3 \times x \times y = 15xy$

(iii) $5x \times (-3y) = 5 \times x \times (-3) \times y$

$= 5 \times (-3) \times x \times y = -15xy$

Notice that all the three products of monomials, $3xy$, $15xy$, $-15xy$, are also monomials.

Some more useful examples follow.

(iv) $5x \times 4x^2 = (5 \times 4) \times (x \times x^2)$

$= 20 \times x^3 = 20x^3$

(v) $5x \times (-4xyz) = (5 \times -4) \times (x \times xyz)$

$= -20 \times (x \times x \times yz) = -20x^2yz$

Note that $5 \times 4 = 20$
i.e., coefficient of product = coefficient of first monomial × coefficient of second monomial;
and $x \times x^2 = x^3$
i.e., algebraic factor of product = algebraic factor of first monomial × algebraic factor of second monomial.

Observe how we collect the powers of different variables in the algebraic parts of the two monomials. While doing so, we use the rules of exponents and powers.

8.3.2 Multiplying three or more monomials

Observe the following examples.

(i) $2x \times 5y \times 7z = (2x \times 5y) \times 7z = 10xy \times 7z = 70xyz$

(ii) $4xy \times 5x^2y^2 \times 6x^3y^3 = (4xy \times 5x^2y^2) \times 6x^3y^3 = 20x^3y^3 \times 6x^3y^3 = 120x^3y^3 \times x^3y^3$

$= 120\,(x^3 \times x^3) \times (y^3 \times y^3) = 120x^6 \times y^6 = 120x^6y^6$

It is clear that we first multiply the first two monomials and then multiply the resulting monomial by the third monomial. This method can be extended to the product of any number of monomials.

TRY THESE

Find $4x \times 5y \times 7z$
First find $4x \times 5y$ and multiply it by $7z$;
or first find $5y \times 7z$ and multiply it by $4x$.
Is the result the same? What do you observe?
Does the order in which you carry out the multiplication matter?

We can find the product in other way also.
$4xy \times 5x^2y^2 \times 6x^3 y^3$
$= (4 \times 5 \times 6) \times (x \times x^2 \times x^3) \times (y \times y^2 \times y^3)$
$= 120\, x^6y^6$

Example 3: Complete the table for area of a rectangle with given length and breadth.

Solution:

length	breadth	area
$3x$	$5y$	$3x \times 5y = 15xy$
$9y$	$4y^2$	....
$4ab$	$5bc$	....
$2l^2m$	$3lm^2$	....

Example 4: Find the volume of each rectangular box with given length, breadth and height.

	length	breadth	height
(i)	$2ax$	$3by$	$5cz$
(ii)	m^2n	n^2p	p^2m
(iii)	$2q$	$4q^2$	$8q^3$

Solution: Volume = length × breadth × height

Hence, for (i) volume $= (2ax) \times (3by) \times (5cz)$
$= 2 \times 3 \times 5 \times (ax) \times (by) \times (cz) = 30abcxyz$

for (ii) volume $= m^2n \times n^2p \times p^2m$
$= (m^2 \times m) \times (n \times n^2) \times (p \times p^2) = m^3n^3p^3$

for (iii) volume $= 2q \times 4q^2 \times 8q^3$
$= 2 \times 4 \times 8 \times q \times q^2 \times q^3 = 64q^6$

EXERCISE 8.2

1. Find the product of the following pairs of monomials.
   (i) $4, 7p$ (ii) $-4p, 7p$ (iii) $-4p, 7pq$ (iv) $4p^3, -3p$
   (v) $4p, 0$
2. Find the areas of rectangles with the following pairs of monomials as their lengths and breadths respectively.
   (p, q); $(10m, 5n)$; $(20x^2, 5y^2)$; $(4x, 3x^2)$; $(3mn, 4np)$

3. Complete the table of products.

First monomial → Second monomial ↓	$2x$	$-5y$	$3x^2$	$-4xy$	$7x^2y$	$-9x^2y^2$
$2x$	$4x^2$	…	…	…	…	…
$-5y$	…	…	$-15x^2y$	…	…	…
$3x^2$	…	…	…	…	…	…
$-4xy$	…	…	…	…	…	…
$7x^2y$	…	…	…	…	…	…
$-9x^2y^2$	…	…	…	…	…	…

4. Obtain the volume of rectangular boxes with the following length, breadth and height respectively.

(i) $5a, 3a^2, 7a^4$ (ii) $2p, 4q, 8r$ (iii) $xy, 2x^2y, 2xy^2$ (iv) $a, 2b, 3c$

5. Obtain the product of

(i) xy, yz, zx (ii) $a, -a^2, a^3$ (iii) $2, 4y, 8y^2, 16y^3$

(iv) $a, 2b, 3c, 6abc$ (v) $m, -mn, mnp$

8.4 Multiplying a Monomial by a Polynomial

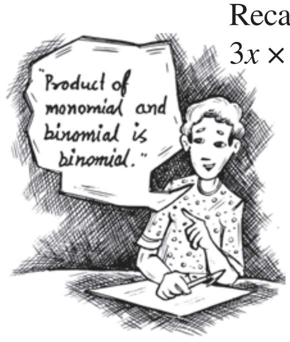
Expression that contains two terms is called a **binomial**. An expression containing three terms is a **trinomial** and so on. In general, an expression containing, one or more terms with non-zero coefficient (with variables having non negative integers as exponents) is called a **polynomial**.

8.4.1 Multiplying a monomial by a binomial

Let us multiply the monomial $3x$ by the binomial $5y + 2$, i.e., find $3x \times (5y + 2) = ?$

Recall that $3x$ and $(5y + 2)$ represent numbers. Therefore, using the distributive law,

$3x \times (5y + 2) = (3x \times 5y) + (3x \times 2) = 15xy + 6x$



We commonly use distributive law in our calculations. For example:

$7 \times 106 = 7 \times (100 + 6)$
$= 7 \times 100 + 7 \times 6$ (Here, we used distributive law)
$= 700 + 42 = 742$

$7 \times 38 = 7 \times (40 - 2)$
$= 7 \times 40 - 7 \times 2$ (Here, we used distributive law)
$= 280 - 14 = 266$

Similarly, $(-3x) \times (-5y + 2) = (-3x) \times (-5y) + (-3x) \times (2) = 15xy - 6x$

and $5xy \times (y^2 + 3) = (5xy \times y^2) + (5xy \times 3) = 5xy^3 + 15xy.$

What about a binomial × monomial? For example, $(5y + 2) \times 3x = ?$

We may use commutative law as : $7 \times 3 = 3 \times 7$; or in general $a \times b = b \times a$

Similarly, $(5y + 2) \times 3x = 3x \times (5y + 2) = 15xy + 6x$ as before.

TRY THESE

Find the product (i) $2x\ (3x + 5xy)$ (ii) $a^2\ (2ab - 5c)$

8.4.2 Multiplying a monomial by a trinomial

Consider $3p \times (4p^2 + 5p + 7)$. As in the earlier case, we use distributive law;

$$3p \times (4p^2 + 5p + 7) = (3p \times 4p^2) + (3p \times 5p) + (3p \times 7)$$
$$= 12p^3 + 15p^2 + 21p$$

Multiply each term of the trinomial by the monomial and add products.

> Observe, by using the distributive law, we are able to carry out the multiplication term by term.

TRY THESE

Find the product:
$(4p^2 + 5p + 7) \times 3p$

Example 5: Simplify the expressions and evaluate them as directed:

(i) $x(x - 3) + 2$ for $x = 1$, (ii) $3y(2y - 7) - 3(y - 4) - 63$ for $y = -2$

Solution:

(i) $x(x - 3) + 2 = x^2 - 3x + 2$

For $x = 1$, $x^2 - 3x + 2 = (1)^2 - 3(1) + 2$
$= 1 - 3 + 2 = 3 - 3 = 0$

(ii) $3y(2y - 7) - 3(y - 4) - 63 = 6y^2 - 21y - 3y + 12 - 63$
$= 6y^2 - 24y - 51$

For $y = -2$, $6y^2 - 24y - 51 = 6(-2)^2 - 24(-2) - 51$
$= 6 \times 4 + 24 \times 2 - 51$
$= 24 + 48 - 51 = 72 - 51 = 21$

Example 6: Add

(i) $5m(3 - m)$ and $6m^2 - 13m$ (ii) $4y(3y^2 + 5y - 7)$ and $2(y^3 - 4y^2 + 5)$

Solution:

(i) First expression $= 5m(3 - m) = (5m \times 3) - (5m \times m) = 15m - 5m^2$

Now adding the second expression to it, $15m - 5m^2 + 6m^2 - 13m = m^2 + 2m$

(ii) The first expression $= 4y(3y^2 + 5y - 7) = (4y \times 3y^2) + (4y \times 5y) + (4y \times (-7))$
$= 12y^3 + 20y^2 - 28y$

The second expression $= 2(y^3 - 4y^2 + 5) = 2y^3 + 2 \times (-4y^2) + 2 \times 5$
$= 2y^3 - 8y^2 + 10$

Adding the two expressions,

$$\begin{array}{rrrrr} & 12y^3 & + & 20y^2 - 28y & \\ + & 2y^3 & - & 8y^2 & + 10 \\ \hline & 14y^3 & + & 12y^2 - 28y & + 10 \end{array}$$

Example 7: Subtract $3pq(p - q)$ from $2pq(p + q)$.

Solution: We have $3pq(p - q) = 3p^2q - 3pq^2$ and

$2pq(p + q) = 2p^2q + 2pq^2$

Subtracting,

$$\begin{array}{rrr} 2p^2q & + & 2pq^2 \\ 3p^2q & - & 3pq^2 \\ - & + & \\ \hline -p^2q & + & 5pq^2 \end{array}$$

EXERCISE 8.3

1. Carry out the multiplication of the expressions in each of the following pairs.

(i) $4p, q + r$ (ii) $ab, a - b$ (iii) $a + b, 7a^2b^2$ (iv) $a^2 - 9, 4a$

(v) $pq + qr + rp, 0$

2. Complete the table.

	First expression	Second expression	Product
(i)	a	$b + c + d$	...
(ii)	$x + y - 5$	$5xy$	...
(iii)	p	$6p^2 - 7p + 5$	...
(iv)	$4p^2q^2$	$p^2 - q^2$	...
(v)	$a + b + c$	abc	...

3. Find the product.

(i) $(a^2) \times (2a^{22}) \times (4a^{26})$ (ii) $\left(\frac{2}{3}xy\right) \times \left(\frac{-9}{10}x^2y^2\right)$

(iii) $\left(-\frac{10}{3}pq^3\right) \times \left(\frac{6}{5}p^3q\right)$ (iv) $x \times x^2 \times x^3 \times x^4$

4. (a) Simplify $3x(4x - 5) + 3$ and find its values for (i) $x = 3$ (ii) $x = \frac{1}{2}$.

(b) Simplify $a(a^2 + a + 1) + 5$ and find its value for (i) $a = 0$, (ii) $a = 1$ (iii) $a = -1$.

5. (a) Add: $p(p - q), q(q - r)$ and $r(r - p)$

(b) Add: $2x(z - x - y)$ and $2y(z - y - x)$

(c) Subtract: $3l(l - 4m + 5n)$ from $4l(10n - 3m + 2l)$

(d) Subtract: $3a(a + b + c) - 2b(a - b + c)$ from $4c(-a + b + c)$

8.5 Multiplying a Polynomial by a Polynomial

8.5.1 Multiplying a binomial by a binomial

Let us multiply one binomial $(2a + 3b)$ by another binomial, say $(3a + 4b)$. We do this step-by-step, as we did in earlier cases, following the distributive law of multiplication,

$$(3a + 4b) \times (2a + 3b) = 3a \times (2a + 3b) + 4b \times (2a + 3b)$$

Observe, every term in one binomial multiplies every term in the other binomial.

$$= (3a \times 2a) + (3a \times 3b) + (4b \times 2a) + (4b \times 3b)$$

$$= 6a^2 + 9ab + 8ba + 12b^2$$

$$= 6a^2 + 17ab + 12b^2 \qquad \text{(Since } ba = ab\text{)}$$

When we carry out term by term multiplication, we expect $2 \times 2 = 4$ terms to be present. But two of these are like terms, which are combined, and hence we get 3 terms. **In multiplication of polynomials with polynomials, we should always look for like terms, if any, and combine them**.

Example 8: Multiply

(i) $(x - 4)$ and $(2x + 3)$ (ii) $(x - y)$ and $(3x + 5y)$

Solution:

(i) $(x - 4) \times (2x + 3) = x \times (2x + 3) - 4 \times (2x + 3)$

$= (x \times 2x) + (x \times 3) - (4 \times 2x) - (4 \times 3) = 2x^2 + 3x - 8x - 12$

$= 2x^2 - 5x - 12$ (Adding like terms)

(ii) $(x - y) \times (3x + 5y) = x \times (3x + 5y) - y \times (3x + 5y)$

$= (x \times 3x) + (x \times 5y) - (y \times 3x) - (y \times 5y)$

$= 3x^2 + 5xy - 3yx - 5y^2 = 3x^2 + 2xy - 5y^2$ (Adding like terms)

Example 9: Multiply

(i) $(a + 7)$ and $(b - 5)$ (ii) $(a^2 + 2b^2)$ and $(5a - 3b)$

Solution:

(i) $(a + 7) \times (b - 5) = a \times (b - 5) + 7 \times (b - 5)$

$= ab - 5a + 7b - 35$

Note that there are no like terms involved in this multiplication.

(ii) $(a^2 + 2b^2) \times (5a - 3b) = a^2 (5a - 3b) + 2b^2 \times (5a - 3b)$

$= 5a^3 - 3a^2b + 10ab^2 - 6b^3$

8.5.2 Multiplying a binomial by a trinomial

In this multiplication, we shall have to multiply each of the three terms in the trinomial by each of the two terms in the binomial. We shall get in all $3 \times 2 = 6$ terms, which may reduce to 5 or less, if the term by term multiplication results in like terms. Consider

$$\underbrace{(a+7)}_{\text{binomial}} \times \underbrace{(a^2+3a+5)}_{\text{trinomial}} = a \times (a^2 + 3a + 5) + 7 \times (a^2 + 3a + 5)$$

[using the distributive law]

$= a^3 + 3a^2 + 5a + 7a^2 + 21a + 35$

$= a^3 + (3a^2 + 7a^2) + (5a + 21a) + 35$

$= a^3 + 10a^2 + 26a + 35$ (Why are there only 4 terms in the final result?)

Example 10: Simplify $(a + b)(2a - 3b + c) - (2a - 3b)c$.

Solution: We have

$(a + b)(2a - 3b + c) = a(2a - 3b + c) + b(2a - 3b + c)$

$= 2a^2 - 3ab + ac + 2ab - 3b^2 + bc$

$= 2a^2 - ab - 3b^2 + bc + ac$ (Note, $-3ab$ and $2ab$ are like terms)

and $(2a - 3b)c = 2ac - 3bc$

Therefore,

$(a + b)(2a - 3b + c) - (2a - 3b)c = 2a^2 - ab - 3b^2 + bc + ac - (2ac - 3bc)$

$= 2a^2 - ab - 3b^2 + bc + ac - 2ac + 3bc$

$= 2a^2 - ab - 3b^2 + (bc + 3bc) + (ac - 2ac)$

$= 2a^2 - 3b^2 - ab + 4bc - ac$

EXERCISE 8.4

1. Multiply the binomials.

(i) $(2x + 5)$ and $(4x - 3)$
(ii) $(y - 8)$ and $(3y - 4)$
(iii) $(2.5l - 0.5m)$ and $(2.5l + 0.5m)$
(iv) $(a + 3b)$ and $(x + 5)$
(v) $(2pq + 3q^2)$ and $(3pq - 2q^2)$
(vi) $\left(\frac{3}{4}a^2 + 3b^2\right)$ and $4\left(a^2 - \frac{2}{3}b^2\right)$

2. Find the product.

(i) $(5 - 2x)(3 + x)$
(ii) $(x + 7y)(7x - y)$
(iii) $(a^2 + b)(a + b^2)$
(iv) $(p^2 - q^2)(2p + q)$

3. Simplify.

(i) $(x^2 - 5)(x + 5) + 25$
(ii) $(a^2 + 5)(b^3 + 3) + 5$
(iii) $(t + s^2)(t^2 - s)$
(iv) $(a + b)(c - d) + (a - b)(c + d) + 2(ac + bd)$
(v) $(x + y)(2x + y) + (x + 2y)(x - y)$
(vi) $(x + y)(x^2 - xy + y^2)$
(vii) $(1.5x - 4y)(1.5x + 4y + 3) - 4.5x + 12y$
(viii) $(a + b + c)(a + b - c)$

WHAT HAVE WE DISCUSSED?

1. Expressions are formed from **variables** and **constants**.
2. Terms are added to form **expressions**. Terms themselves are formed as product of **factors**.
3. Expressions that contain exactly one, two and three terms are called **monomials**, **binomials** and **trinomials** respectively. In general, any expression containing one or more terms with non-zero coefficients (and with variables having non- negative integers as exponents) is called a **polynomial**.
4. **Like** terms are formed from the same variables and the powers of these variables are the same, too. Coefficients of like terms need not be the same.
5. While adding (or subtracting) polynomials, first look for like terms and add (or subtract) them; then handle the unlike terms.
6. There are number of situations in which we need to multiply algebraic expressions: for example, in finding area of a rectangle, the sides of which are given as expressions.
7. A monomial multiplied by a monomial always gives a monomial.
8. While multiplying a polynomial by a monomial, we multiply every term in the polynomial by the monomial.
9. In carrying out the multiplication of a polynomial by a binomial (or trinomial), we multiply term by term, i.e., every term of the polynomial is multiplied by every term in the binomial (or trinomial). Note that in such multiplication, we may get terms in the product which are like and have to be combined.